Input-Output Relations in Optical Cavities: a Simple Point of View

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Abstract

In this work we present a very simple approach to input-output relations in optical cavities, limiting ourselves to one- and two-photon states of the field. After field quantization, we derive the non-unitary transformation between *Inside* and *Outside* annihilation and creation operators. Then we express the most general two-photon state generated by *Inside* creation operators, through base states generated by *Outside* creation operators. After renormalization of coefficients of inside two-photon state, we calculate the outside photon-number probability distribution in a general case. Finally we treat with some detail the single mode and symmetrical cavity case.

1 Introduction

The problem of interaction between a pair of atoms in free space and in a cavity has been subject of several investigations in the past years [1, 2, 3, 4, 7, 8]. Recently the spontaneous emission of a pair of two identical atoms or molecules in a planar Fabry-Pérot microcavity has been subject of theoretical and experimental research [9, 10]. This work starts from our attempt to give a simple interpretation to some recent experimental results. Consider a process of spontaneous emission of a pair of photons by two distinct molecules inside a microcavity. If we measure the number of photons emitted outside the cavity, and if there are not dissipative phenomena, we will find only three possible results: two photons detected on the right and no one on the left, two photons detected on the left and no one on the right, one photon detected on the left and one on the right. In this work we look for a solution to this question: if we know the distribution of the number of photons outside the cavity, how can we obtain information about the process of emission which generated the photons without completely solving the problem? Our basic idea is to describe the electromagnetic field inside the cavity by means of a two-photon state as general as possible; the coefficients of the expansion of this state in a proper basis set depend on the process which generated the state itself. If we project this state on the basis number-states defined outside the cavity, we obtain directly the probability distribution we look for. Therefore if we change the above mentioned coefficients, we can study how the probability distribution changes, and by comparison with the measured distribution we can obtain the values of these coefficients getting information about the process active inside the cavity. This procedure is not orthodox or fully justified, however the obtained results are in qualitative agreement with experimental results.

The relations of Input-Output and their connection with the traditional stochastic methods based on Langevin equations have been studied by Knöll *et al.* [22]. The aim of the present paper is to generalize the methods used to describe an optical element with two inputs and two outputs, as a Beam-Splitter, to derive simple relations between fields inside and outside a planar Fabry-Pérot cavity. In Sec. II we quantize the electromagnetic field generalizing to three dimensions an approach presented by Barnett *et al.* [11] for one-dimensional fields. In Sec. III we restrict our attention to one-dimensional fields and derive the relations between operators defined in the space Inside and Outside the cavity. Then, in Sec. IV we build and study the states of the field generated by linear and bilinear forms of creation operators both inside and outside the cavity and, in Sec. V, we calculate the outside probability distributions for these two-photon states. Finally, we summarize our results in Sec. VI.

2 Spatial modes and Field Quantization

Now we calculate the appropriate normal modes for quantization of the electromagnetic field. In a traditional approach, one first determines the modes of the classical boundary value problem, then one quantizes the field in terms of these modes [13, 14]. An alternative approach has been presented by Barnett *et al.* [11]. We have generalized this work, restricted to an one-dimensional fields, to three-dimensional fields.

In free space, having fixed the Coulomb gauge, the spatial variations of the electromagnetic field can be described by the solutions $\mathbf{u}_k(\mathbf{r})$ of the Helmholtz equations

$$\nabla^2 \mathbf{u}_k(\mathbf{r}) + k^2 \mathbf{u}_k(\mathbf{r}) = 0. \tag{1}$$

For each value of k, there are two solutions that we indicate with $\mathbf{u}_{R\lambda}(\mathbf{k}, \mathbf{r})$ and $\mathbf{u}_{L\lambda}(\mathbf{k}, \mathbf{r})$. We choose them as plane waves that are incident respectively from left and right on the plane of equation $z = \text{const.}$, as shown in Fig. 1.

Here $\lambda = 1, 2$ is the polarization index and $\mathbf{k}$ is the wave-vector such as $|\mathbf{k}|^2 = k^2 \equiv \omega^2/c^2$. The explicit form of $\mathbf{u}_{R\lambda}(\mathbf{k}, \mathbf{r})$ and $\mathbf{u}_{L\lambda}(\mathbf{k}, \mathbf{r})$ is therefore

$$\begin{aligned} \mathbf{u}_{R\lambda}(\mathbf{k},\, \mathbf{r}) &= \boldsymbol{\epsilon}_\lambda(\mathbf{k}_+) \exp(i\mathbf{k}_+ \cdot \mathbf{r}), \\ \mathbf{u}_{L\lambda}(\mathbf{k},\, \mathbf{r}) &= \boldsymbol{\epsilon}_\lambda(\mathbf{k}_-) \exp(i\mathbf{k}_- \cdot \mathbf{r}), \end{aligned} \tag{2}$$

where we have defined

$$\mathbf{k}_\pm = k(\sin\theta\, \cos\phi,\ \sin\theta\, \sin\phi,\ \pm\cos\theta), \tag{3}$$

for $(0 \leq \theta \leq \pi/2)$, and

$$\begin{aligned} \boldsymbol{\epsilon}_1(\mathbf{k}_\pm) &= (\sin\phi, -\cos\phi, 0), \\ \boldsymbol{\epsilon}_2(\mathbf{k}_\pm) &= (\cos\theta\cos\phi, \cos\theta\sin\phi, \mp\sin\theta). \end{aligned} \tag{4}$$

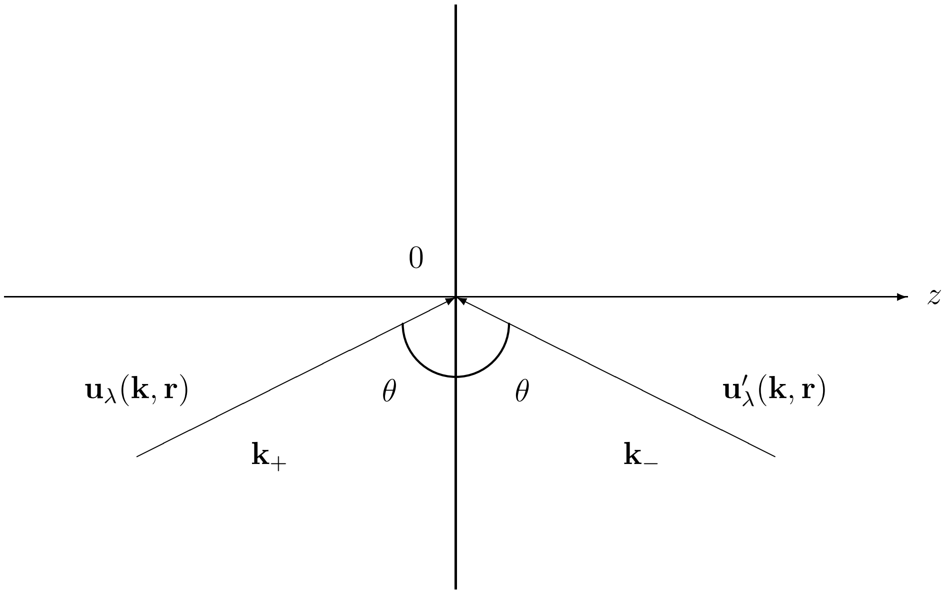

FIG.1. Scheme of the plane waves $\mathbf{u}_\lambda(\mathbf{k},\mathbf{r})$ and $\mathbf{u}'_\lambda(\mathbf{k},\mathbf{r})$ incident respectively from left and from right on the plane $z=0$.

Now, following a standard procedure [15, 16], we introduce the mode creation and destruction operators $\hat{a}_{R\lambda}(\mathbf{k})$ and $\hat{a}_{L\lambda}(\mathbf{k})$ as

$$\begin{aligned} &\hat{a}^\dagger_{R\lambda}(\mathbf{k}) \quad \text{and} \quad \hat{a}_{R\lambda}(\mathbf{k}) \quad \text{for the mode} \quad \mathbf{u}_{R\lambda}(\mathbf{k},\mathbf{r}), \\ &\hat{a}^\dagger_{L\lambda}(\mathbf{k}) \quad \text{and} \quad \hat{a}_{L\lambda}(\mathbf{k}) \quad \text{for the mode} \quad \mathbf{u}_{L\lambda}(\mathbf{k},\mathbf{r}). \end{aligned} \tag{5}$$

Being $\mathbf{k}$ a three-dimensional continuous variable, these operators satisfy the following commutation relations:

$$\begin{aligned} \left[\hat{a}_{R\lambda}(\mathbf{k}),\, \hat{a}^\dagger_{R\lambda'}(\mathbf{k}')\right] &= \left[\hat{a}_{L\lambda}(\mathbf{k}),\, \hat{a}^\dagger_{L\lambda'}(\mathbf{k}')\right] = \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}'), \\ \left[\hat{a}_{L\lambda}(\mathbf{k}),\, \hat{a}^\dagger_{R\lambda'}(\mathbf{k}')\right] &= \left[\hat{a}_{R\lambda}(\mathbf{k}),\, \hat{a}^\dagger_{L\lambda'}(\mathbf{k}')\right] = 0. \end{aligned} \tag{6}$$

The vector potential operator is written, in Heisenberg representation, as

$$\hat{\mathbf{A}}(\mathbf{r},t) = \hat{\mathbf{A}}^+(\mathbf{r},t) + \hat{\mathbf{A}}^-(\mathbf{r},t), \tag{7}$$

where

$$
\begin{aligned}
\hat{\mathbf{A}}^{+}(\mathbf{r},t) = \int d\mathbf{k} \Bigg\{ \left(\frac{\hbar}{16\pi^3 \varepsilon_0 \omega} \right)^{1/2} \\
\times \sum_{\lambda=1,2} \Big[\mathbf{u}_{R\lambda}(\mathbf{k},\, \mathbf{r}) \hat{a}_{R\lambda}(\mathbf{k}) + \mathbf{u}_{L\lambda}(\mathbf{k},\, \mathbf{r}) \hat{a}_{L\lambda}(\mathbf{k}) \Big] \exp(-i\omega t) \Bigg\} ,
\end{aligned}
\tag{8}
$$

and $\hat{\mathbf{A}}^{-}(\mathbf{r},t) = [\hat{\mathbf{A}}^{+}(\mathbf{r},t)]^{\dagger}$.
Now we consider an infinitesimally thin non-absorbing dielectric slab, placed in the plane $z = z_0$ and represented by a non-homogeneous dielectric constant of equation

$$
\varepsilon(z) = \varepsilon_0 \left[\eta \delta(z - z_0) + 1 \right] , \tag{9}
$$

where η is a constant depending upon the optical properties of dielectric media [17].

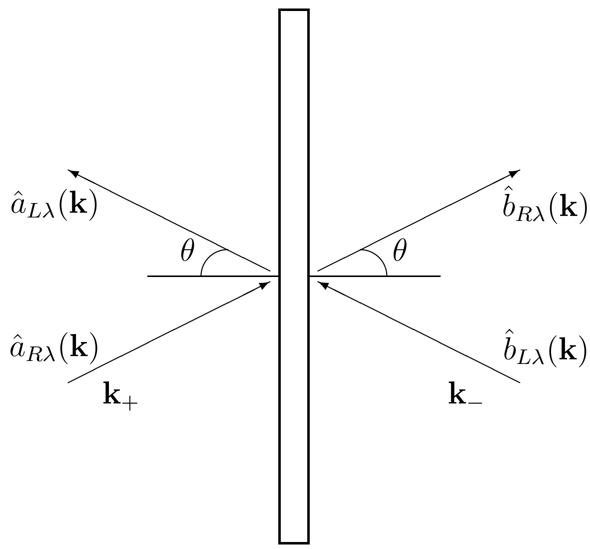

FIG. 2. Schematic representation of the dielectric slab that transform *Input* operators $\hat{a}_{R\lambda}(\mathbf{k})$ and $\hat{b}_{L\lambda}(\mathbf{k})$ in *Output* operators $\hat{a}_{L\lambda}(\mathbf{k})$ and $\hat{b}_{R\lambda}(\mathbf{k})$.

The complex reflection and transmission coefficients $r_\lambda(\mathbf{k})$ and $t_\lambda(\mathbf{k})$ of the slab depend upon the incident radiation wave-vector and polarization and they have the following unitary lossless properties, valid for all values of $\mathbf{k}$, λ:

$$
\begin{aligned}
|r_\lambda(\mathbf{k})|^2 + |t_\lambda(\mathbf{k})|^2 &= 1, \\
t_\lambda(\mathbf{k}) r^*_\lambda(\mathbf{k}) + r_\lambda(\mathbf{k}) t^*_\lambda(\mathbf{k}) &= 0.
\end{aligned}
\tag{10}
$$

Consider the arrangement in Fig. 2, which shows the interaction of two fields of wavevectors $\mathbf{k}_+$ and $\mathbf{k}_-$ respectively, on a dielectric slab placed at $z = 0$. It is well known [18, 19] that the annihilation operators $\hat{a}_{L\lambda}(\mathbf{k})$ and $\hat{c}_{R\lambda}(\mathbf{k})$ associated to Output modes are related to the annihilation operators $\hat{a}_{R\lambda}(\mathbf{k})$ and $\hat{c}_{L\lambda}(\mathbf{k})$ associated to Input modes, by the 2×2 matrix

$$
\begin{pmatrix} \hat{a}_{L\lambda}(\mathbf{k}) \\ \hat{c}_{R\lambda}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} r_\lambda(\mathbf{k}) & t_\lambda(\mathbf{k}) \\ t_\lambda(\mathbf{k}) & r_\lambda(\mathbf{k}) \end{pmatrix} \begin{pmatrix} \hat{a}_{R\lambda}(\mathbf{k}) \\ \hat{c}_{L\lambda}(\mathbf{k}) \end{pmatrix}, \tag{11}
$$

whose unitarity is assured by Eqs. (10). Now imagine to put a pair of infinitesimally thin dielectric slabs at $z = \pm l/2$, to form a planar Fabry-Pérot cavity, as shown in Fig. 3.
We impose boundary conditions of the form (11) both on slab 1 and on slab 2 obtaining respectively,

$$
\begin{aligned}
\hat{b}_{R\lambda}(\mathbf{k}) e^{i\varphi_R(-l/2)} &= t_{1\lambda}(\mathbf{k}) \hat{a}_{R\lambda}(\mathbf{k}) e^{i\varphi_R(-l/2)} + r_{1\lambda}(\mathbf{k}) \hat{b}_{L\lambda}(\mathbf{k}) e^{i\varphi_L(-l/2)}, \\
\hat{a}_{L\lambda}(\mathbf{k}) e^{i\varphi_L(-l/2)} &= t_{1\lambda}(\mathbf{k}) \hat{b}_{L\lambda}(\mathbf{k}) e^{i\varphi_L(-l/2)} + r_{1\lambda}(\mathbf{k}) \hat{a}_{R\lambda}(\mathbf{k}) e^{i\varphi_R(-l/2)},
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
\hat{c}_{R\lambda}(\mathbf{k}) e^{i\varphi_R(l/2)} &= t_{2\lambda}(\mathbf{k}) \hat{b}_{R\lambda}(\mathbf{k}) e^{i\varphi_R(l/2)} + r_{2\lambda}(\mathbf{k}) \hat{c}_{L\lambda}(\mathbf{k}) e^{i\varphi_L(l/2)}, \\
\hat{b}_{L\lambda}(\mathbf{k}) e^{i\varphi_L(l/2)} &= t_{2\lambda}(\mathbf{k}) \hat{c}_{L\lambda}(\mathbf{k}) e^{i\varphi_L(l/2)} + r_{2\lambda}(\mathbf{k}) \hat{b}_{R\lambda}(\mathbf{k}) e^{i\varphi_R(l/2)},
\end{aligned}
\tag{13}
$$

where $\varphi_R(z)$ and $\varphi_L(z)$ are the phases generated by field propagation inside the cavity.

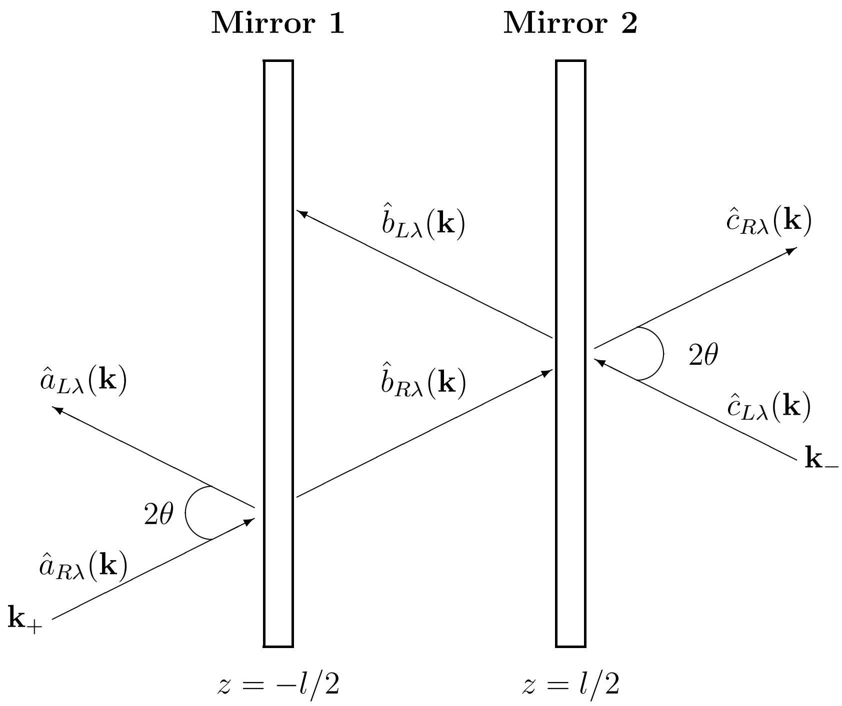

FIG. 3. Schematic representation of a planar Fabry-Pérot cavity with notation for *Input* operators $\hat{a}_{R\lambda}(\mathbf{k})$, $\hat{c}_{L\lambda}(\mathbf{k})$, *Inside* operators $\hat{b}_{R\lambda}(\mathbf{k})$, $\hat{b}_{L\lambda}(\mathbf{k})$ and *Outside* operators $\hat{a}_{L\lambda}(\mathbf{k})$, $\hat{c}_{R\lambda}(\mathbf{k})$.

We freely choose the zero phases in the middle of the cavity, so that $\varphi_F(-z) = -\varphi_F(z)$, where $F = R$, L and $\varphi_R(z) = -\varphi_L(z)$. Then we can put

$$\varphi_L(\pm l/2) = \mp\delta/2, \qquad \varphi_R(\pm l/2) = \pm\delta/2, \tag{14}$$

where

$$\delta \equiv \frac{\omega}{c} l \cos\theta, \tag{15}$$

is half of the phase gained in a double traversal of the cavity [20]. From the first of Eqs. (12) and the second of (13), we can express the intracavity operators $\hat{b}_{R\lambda}(\mathbf{k})$ and $\hat{b}_{L\lambda}(\mathbf{k})$ as

$$\hat{b}_{R\lambda}(\mathbf{k}) = \hat{a}_{R\lambda}(\mathbf{k}) I_\lambda(\mathbf{k}) + \hat{c}_{L\lambda}(\mathbf{k}) J'_\lambda(\mathbf{k}), \tag{16}$$

$$\hat{b}_{L\lambda}(\mathbf{k}) = \hat{c}_{L\lambda}(\mathbf{k}) J_\lambda(\mathbf{k}) + \hat{a}_{R\lambda}(\mathbf{k}) I'_\lambda(\mathbf{k}). \tag{17}$$

where we have defined, following Ref. [13],

$$D_\lambda(\mathbf{k}) \equiv 1 - r_{1\lambda}(\mathbf{k}) r_{2\lambda}(\mathbf{k}) \exp(2i\delta), \tag{18}$$

$$\begin{aligned} I_\lambda(\mathbf{k}) &= t_{1\lambda}(\mathbf{k})/D_\lambda(\mathbf{k}), \\ I'_\lambda(\mathbf{k}) &= t_{2\lambda}(\mathbf{k})/D_\lambda(\mathbf{k}), \end{aligned} \tag{19}$$

$$\begin{aligned} J_\lambda(\mathbf{k}) &= r_{2\lambda}(\mathbf{k}) \exp(i\delta) I_\lambda(\mathbf{k}), \\ J'_\lambda(\mathbf{k}) &= r_{1\lambda}(\mathbf{k}) \exp(i\delta) I'_\lambda(\mathbf{k}). \end{aligned} \tag{20}$$

Substituting Eqs. (16)-(17) in the second of the (12) and in the first of (13), we obtain

$$\hat{c}_{R\lambda}(\mathbf{k}) = \hat{a}_{R\lambda}(\mathbf{k}) T_\lambda(\mathbf{k}) + \hat{a}_{L\lambda}(\mathbf{k}) R'_\lambda(\mathbf{k}), \tag{21}$$

$$\hat{a}_{L\lambda}(\mathbf{k}) = \hat{c}_{L\lambda}(\mathbf{k}) R_\lambda(\mathbf{k}) + \hat{a}_{R\lambda}(\mathbf{k}) T'_\lambda(\mathbf{k}), \tag{22}$$

where

$$T_\lambda(\mathbf{k}) = T'_\lambda(\mathbf{k}) \equiv \frac{t_{1\lambda}(\mathbf{k}) t_{2\lambda}(\mathbf{k})}{D_\lambda(\mathbf{k})}, \tag{23}$$

$$R_\lambda(\mathbf{k}) \equiv \frac{r_{1\lambda}(\mathbf{k}) \exp(-i\delta) + r_{2\lambda}(\mathbf{k}) \left[t^2_{1\lambda}(\mathbf{k}) - r^2_{1\lambda}(\mathbf{k})\right] \exp(i\delta)}{D_\lambda(\mathbf{k})}, \tag{24}$$

$$R'_\lambda(\mathbf{k}) \equiv \frac{r_{2\lambda}(\mathbf{k}) \exp(-i\delta) + r_{1\lambda}(\mathbf{k}) \left[t^2_{2\lambda}(\mathbf{k}) - r^2_{2\lambda}(\mathbf{k})\right] \exp(i\delta)}{D_\lambda(\mathbf{k})}. \tag{25}$$

The coefficients $R_\lambda(\mathbf{k})$ and $T_\lambda(\mathbf{k})$ and the corresponding primed represent the reflection and transmission coefficients of a Fabry-Pérot cavity as a whole. It can readily be shown that they satisfy the conditions [13]

$$|R_\lambda(\mathbf{k})| = |R'_\lambda(\mathbf{k})|\,, \tag{26}$$

$$|R_\lambda(\mathbf{k})|^2 + |T_\lambda(\mathbf{k})|^2 = |R'_\lambda(\mathbf{k})|^2 + |T'_\lambda(\mathbf{k})|^2 = 1, \tag{27}$$

$$R^*_\lambda(\mathbf{k})T'_\lambda(\mathbf{k}) + R'_\lambda(\mathbf{k})T^*_\lambda(\mathbf{k}) = 0. \tag{28}$$

Using Eqs. (6) and (26)-(28) it is not difficult to show that the Output operators $\hat{c}_{R\lambda}(\mathbf{k})$ and $\hat{a}_{L\lambda}(\mathbf{k})$ satisfy canonical commutation rules

$$\begin{aligned}
\left[\hat{a}_{L\lambda}(\mathbf{k}),\, \hat{a}^\dagger_{L\lambda'}(\mathbf{k}')\right] &= \left[\hat{c}_{R\lambda}(\mathbf{k}),\, \hat{c}^\dagger_{R\lambda'}(\mathbf{k}')\right] = \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}') \\
\left[\hat{a}_{L\lambda}(\mathbf{k}),\, \hat{c}^\dagger_{R\lambda'}(\mathbf{k}')\right] &= \left[\hat{c}_{R\lambda}(\mathbf{k}),\, \hat{a}^\dagger_{L\lambda'}(\mathbf{k}')\right] = 0,
\end{aligned} \tag{29}$$

while the intracavity operators $\hat{b}_{R\lambda}(\mathbf{k})$ and $\hat{b}_{L\lambda}(\mathbf{k})$ satisfy anomalous commutation rules [12, 11]

$$\begin{aligned}
\left[\hat{b}_{R\lambda}(\mathbf{k}),\, \hat{b}^\dagger_{R\lambda'}(\mathbf{k}')\right] &= \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}')\frac{1-|r_{1\lambda}(\mathbf{k})r_{2\lambda}(\mathbf{k})|^2}{|1-r_{1\lambda}(\mathbf{k})r_{2\lambda}(\mathbf{k})e^{2i\delta}|^2} \\
&= \left[\hat{b}_{L\lambda}(\mathbf{k}),\, \hat{b}^\dagger_{L\lambda'}(\mathbf{k}')\right],
\end{aligned} \tag{30}$$

$$\begin{aligned}
\left[\hat{b}_{L\lambda}(\mathbf{k}),\, \hat{b}^\dagger_{R\lambda'}(\mathbf{k}')\right] &= \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}')\frac{r_{2\lambda}(\mathbf{k})e^{i\delta}[1-|r_{1\lambda}(\mathbf{k})|^2] + r^*_{1\lambda}(\mathbf{k})e^{-i\delta}[1-|r_{2\lambda}(\mathbf{k})|^2]}{|1-r_{1\lambda}(\mathbf{k})r_{2\lambda}(\mathbf{k})e^{2i\delta}|^2} \\
&= \left[\hat{b}_{R\lambda}(\mathbf{k}),\, \hat{b}^\dagger_{L\lambda'}(\mathbf{k}')\right]^*.
\end{aligned} \tag{31}$$

These equations are the three-dimensional generalization of Eqs. (9) given in Ref.

[11] for one-dimensional fields.
Because of the presence of the cavity, the vector potential is now written as

$$\hat{\mathbf{A}}^{+}(\mathbf{r},t) = \int d\mathbf{k} \left(\frac{\hbar}{16\pi^3\varepsilon_0\omega}\right)^{1/2} \sum_{\lambda=1,2} \mathbf{F}_\lambda(\mathbf{k},\mathbf{r}) \exp(-i\omega t), \tag{32}$$

where

$$\mathbf{F}_\lambda(\mathbf{k},\mathbf{r}) = \begin{cases} \hat{a}_{R\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_+)e^{i\mathbf{k}_+\cdot\mathbf{r}} + \hat{a}_{L\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_-)e^{i\mathbf{k}_-\cdot\mathbf{r}}\,, & -\infty < z < l/2\,, \\ \hat{b}_{R\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_+)e^{i\mathbf{k}_+\cdot\mathbf{r}} + \hat{b}_{L\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_-)e^{i\mathbf{k}_-\cdot\mathbf{r}}\,, & -l/2 < z < l/2\,, \\ \hat{c}_{R\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_+)e^{i\mathbf{k}_+\cdot\mathbf{r}} + \hat{c}_{L\lambda}(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k}_-)e^{i\mathbf{k}_-\cdot\mathbf{r}}\,, & l/2 < z < \infty\,. \end{cases} \tag{33}$$

We note that using different pairs of annihilation operators for each region of space delimited by the cavity, as in Eqs. (32-33), we obtain a free-field like representation for all space inside and outside the cavity, but the canonical commutation rules are lost for intracavity operators, as shown by Eqs. (30-31). Conversely if we choose the mode functions for example as in Ref. [14], we lost free-field like representation, but we obtain canonical commutation rules for annihilation and creation operators in whole space. It is easy to show that our result agrees with that of Ref. [14], indeed defining

$$\begin{aligned} \hat{a}_{R\lambda}(\mathbf{k}) &\equiv \hat{a}_{\mathbf{k}\lambda}, \\ \hat{c}_{L\lambda}(\mathbf{k}) &\equiv \hat{a}'_{\mathbf{k}\lambda}, \end{aligned} \tag{34}$$

and substituting Eqs. (16-17) and (21-22) into Eqs. (33), by a straightforward calculation, we obtain

$$\mathbf{F}_\lambda(\mathbf{k},\mathbf{r}) = \boldsymbol{\epsilon}(\mathbf{k},\lambda)\left(U_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} + U'_{\mathbf{k}\lambda}\hat{a}'_{\mathbf{k}\lambda}\right), \qquad -\infty < z < +\infty, \tag{35}$$

where now the mode function $U_{\mathbf{k}\lambda}$ are defined differently on the three regions of the space, as shown in the tables (2.6) and (2.7) of Ref. [14].

3 One-dimensional formulation

In this work we are interested to derive Input-Output relations for a single transverse mode of the cavity, having finite cross-section area $\mathcal{A}$ orthogonal to the z axis which, in fact, depend upon the geometrical and transmitting properties of the cavity itself [5, 6]. Then, following Ref. [21], we impose periodic boundary conditions on both directions x and y, so that the corresponding components of the wave-vector are restricted to the discrete values

$$k_x = \frac{2\pi}{\mathcal{A}^{1/2}} n_x, \quad k_y = \frac{2\pi}{\mathcal{A}^{1/2}} n_y, \quad \text{where} \quad n_x, n_y = 0, \pm 1, \pm 2, \ldots \tag{36}$$

while k_z persists to be continuous positive variable. Then the following conversions are required:

$$\int d\mathbf{k} \to \frac{(2\pi)^2}{\mathcal{A}} \sum_{k_x, k_y} \int_0^\infty dk_z, \tag{37}$$

$$\delta^3(\mathbf{k} - \mathbf{k}') \to \frac{\mathcal{A}}{(2\pi)^2} \delta_{\mathbf{n}\mathbf{n}'} \delta(k_z - k_z'), \tag{38}$$

$$\begin{aligned} \hat{a}_{R\lambda}(\mathbf{k}) &\to (2\pi/\mathcal{A}^{1/2})^{-1} \hat{a}_{R\lambda}(2\pi/\mathcal{A}^{1/2}\,\mathbf{n}, k_z), \\ \hat{c}_{L\lambda}(\mathbf{k}) &\to (2\pi/\mathcal{A}^{1/2})^{-1} \hat{c}_{L\lambda}(2\pi/\mathcal{A}^{1/2}\,\mathbf{n}, k_z), \end{aligned} \tag{39}$$

where $\mathbf{n} \equiv (n_x, n_y)$. Now we fix a linear polarization parallel to the x axis, consider only field excitations with $n_x = n_y = 0$ and define

$$\begin{aligned} \hat{a}_{R\lambda}(2\pi/\mathcal{A}^{1/2}\,\mathbf{n}, k_z)\Big|_{\mathbf{n}=\mathbf{0}} &\equiv c^{1/2} \hat{a}_R(\omega), \\ \hat{c}_{L\lambda}(2\pi/\mathcal{A}^{1/2}\,\mathbf{n}, k_z)\Big|_{\mathbf{n}=\mathbf{0}} &\equiv c^{1/2} \hat{c}_L(\omega), \end{aligned} \tag{40}$$

where we have defined $k_z \equiv k \equiv \omega/c$. These operators have the commutators

$$\begin{aligned} \left[\hat{a}_R(\omega),\, \hat{a}_R^\dagger(\omega')\right] &= \left[\hat{c}_L(\omega),\, \hat{c}_L^\dagger(\omega')\right] = \delta(\omega - \omega'), \\ \left[\hat{a}_R(\omega),\, \hat{c}_L^\dagger(\omega')\right] &= \left[\hat{c}_L(\omega),\, \hat{a}_R^\dagger(\omega')\right] = 0. \end{aligned} \tag{41}$$

Until now, we have limited our attention only to incident operators, however repeating the same procedure for operators $\hat{b}_{R\lambda}(\mathbf{k})$, $\hat{b}_{R\lambda}(\mathbf{k})$ and $\hat{a}_{L\lambda}(\mathbf{k})$, $\hat{c}_{R\lambda}(\mathbf{k})$, we define straightforwardly the corresponding one-dimensional operators $\hat{b}_R(\omega)$, $\hat{b}_L(\omega)$ and $\hat{a}_L(\omega)$, $\hat{c}_R(\omega)$.

Now we assemble the six operators $\hat{a}_R(\omega)$, $\hat{c}_L(\omega)$, $\hat{b}_R(\omega)$, $\hat{b}_L(\omega)$ and $\hat{a}_L(\omega)$, $\hat{c}_R(\omega)$ in three doublets defining

$$\hat{\mathbf{a}}(\omega) = \begin{pmatrix} \hat{a}_1(\omega) \\ \hat{a}_2(\omega) \end{pmatrix} \equiv \begin{pmatrix} \hat{a}_R(\omega) \\ \hat{c}_L(\omega) \end{pmatrix}, \tag{42}$$

$$\hat{\mathbf{b}}(\omega) = \begin{pmatrix} \hat{b}_1(\omega) \\ \hat{b}_2(\omega) \end{pmatrix} \equiv \begin{pmatrix} \hat{b}_R(\omega) \\ \hat{b}_L(\omega) \end{pmatrix}, \tag{43}$$

$$\hat{\mathbf{c}}(\omega) = \begin{pmatrix} \hat{c}_1(\omega) \\ \hat{c}_2(\omega) \end{pmatrix} \equiv \begin{pmatrix} \hat{c}_R(\omega) \\ \hat{a}_L(\omega) \end{pmatrix}. \tag{44}$$

Their geometric meaning is illustrated in Fig. 4.

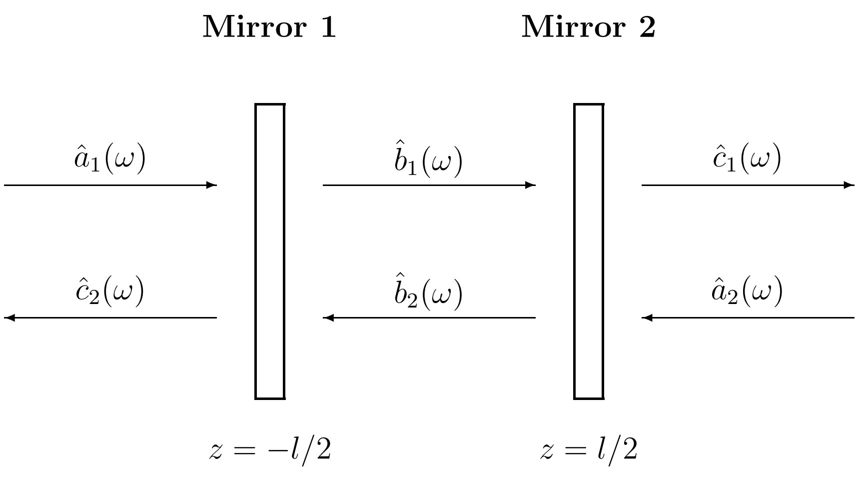

FIG. 4. Schematic representation of a one-dimensional Fabry-Pérot cavity with notation for *Input* operators $\hat{a}_i(\omega)$, $i = 1, 2$, *Inside* operators $\hat{b}_i(\omega)$, $i = 1, 2$ and *Outside* operators $\hat{c}_i(\omega)$, $i = 1, 2$.

Using Eqs. (16-17) in one-dimensional form, it is not difficult to show with a straightforward calculation, that the *Inside* operators $\hat{b}_i(\omega),\ i = 1, 2$ are related to *Incident* operators $\hat{a}_i(\omega),\ i = 1, 2$ by the relations

$$\hat{\mathbf{b}}(\omega) = \mathcal{B}(\omega)\hat{\mathbf{a}}(\omega), \tag{45}$$

where the matrix elements $B_{ij}(\omega)$ of $\mathcal{B}(\omega)$ are given by:

$$\begin{cases} B_{ii}(\omega) = t_i(\omega)/D(\omega), \\ \\ B_{ij}(\omega)\Big|_{i\neq j} = r_i(\omega)\exp(i\omega l/c)B_{jj}(\omega), \end{cases} \qquad i = 1, 2,\ j = 1, 2\ , \tag{46}$$

where

$$D(\omega) \equiv 1 - r_1(\omega)r_2(\omega)\exp(2i\omega l/c). \tag{47}$$

Similarly, the *Outside* operators $\hat{c}_i(\omega)$, $i = 1, 2$ are related to the *Incident* operators by

$$\hat{\mathbf{c}}(\omega) = \mathcal{C}(\omega)\hat{\mathbf{a}}(\omega), \tag{48}$$

where the matrix elements $C_{ij}(\omega)$ $(i, j = 1, 2)$ of $\mathcal{C}(\omega)$ are given by:

$$\begin{cases} C_{ii}(\omega) = t_1(\omega)t_2(\omega)/D(\omega), \\ \\ C_{ij}(\omega)\Big|_{i\neq j} = \dfrac{r_j(\omega)\exp(-i\omega l/c) + r_i(\omega)\exp[i\omega l/c + 2i\arg t_j(\omega)]}{D(\omega)}. \end{cases} \tag{49}$$

Using Eqs. (10)-(49) it is easy to show that the matrix $\mathcal{C}(\omega)$ is unitary:

$$\mathcal{C}(\omega)\cdot\mathcal{C}^{\dagger}(\omega) = \mathcal{C}^{\dagger}(\omega)\cdot\mathcal{C}(\omega) = \mathbb{I}. \tag{50}$$

This leads, with Eqs. (41)-(48), to the result

$$\left[\hat{a}_i(\omega), \hat{a}_j^\dagger(\omega')\right] = \left[\hat{c}_i(\omega), \hat{c}_j^\dagger(\omega')\right] = \delta_{ij}\delta(\omega - \omega'), \qquad i,j = 1,2. \tag{51}$$

For *Inside* operators the commutation rules are found to be

$$\begin{aligned}\left[\hat{b}_i(\omega), \hat{b}_j^\dagger(\omega')\right] &= \left[\mathcal{B}(\omega)\mathcal{B}^\dagger(\omega')\right]_{ij}\delta(\omega - \omega') \\ &\equiv G_{ij}(\omega)\delta(\omega - \omega'),\end{aligned} \tag{52}$$

where, for construction, $\mathcal{G}(\omega) = \mathcal{G}^\dagger(\omega)$, being $G_{ij}(\omega) \equiv [\mathcal{G}(\omega)]_{ij}$, and

$$\begin{cases} G_{11}(\omega) = \dfrac{1 - |r_1(\omega)r_2(\omega)|^2}{D(\omega)} = G_{22}(\omega), \\ \\ G_{12}(\omega) = \dfrac{r_2(\omega)\exp(i\omega l/c)[1 - |r_1(\omega)|^2] + r_1^*(\omega)\exp(-i\omega l/c)[1 - |r_2(\omega)|^2]}{D(\omega)}. \end{cases} \tag{53}$$

Noting that $\mathrm{Det}[\mathcal{B}(\omega)] \neq 0$ for $t_1(\omega) \neq 0$ and $t_2(\omega) \neq 0$, we can invert it to express the relation between Inside and Outside operators as

$$\begin{aligned}\hat{\mathbf{c}}(\omega) &= \mathcal{C}(\omega)\mathcal{B}^{-1}(\omega)\hat{\mathbf{b}}(\omega) \\ &\equiv \mathcal{M}(\omega)\hat{\mathbf{b}}(\omega),\end{aligned} \tag{54}$$

where

$$\mathcal{M}(\omega) = \begin{pmatrix} \dfrac{1}{t_2^*(\omega)} & \dfrac{r_2(\omega)}{t_2(\omega)}\exp(-i\omega l/c) \\ \\ \dfrac{r_1(\omega)}{t_1(\omega)}\exp(-i\omega l/c) & \dfrac{1}{t_1^*(\omega)} \end{pmatrix}. \tag{55}$$

It should be noted that the generic 2×2 matrix that represents a Beam-Splitter must be **unitary** to preserve the canonical bosonic relations of commutation for both *Input* and *Output* operators; in our case $\mathcal{M}(\omega)$ is not unitary at all. In fact, while *Output* operators $\hat{c}_i(\omega)$ satisfy the canonical relations (51), *Input* operators $\hat{b}_i(\omega)$ satisfy the relations (52), that is the so-called anomalous relations of commutation. Using Eqs. (54-55) we obtain

$$\mathcal{M}^\dagger(\omega)\mathcal{M}(\omega) = \mathcal{G}^{-1}(\omega), \tag{56}$$

or, equivalently,

$$\mathcal{M}(\omega)\mathcal{G}(\omega)\mathcal{M}^\dagger(\omega) = \mathbb{I}. \tag{57}$$

Because of the non-unitarity of $\mathcal{M}(\omega)$, the photon-number operator is not conserved on a single mode. In fact from Eq.(54-56) we obtain

$$\hat{\mathbf{c}}^\dagger(\omega) \cdot \hat{\mathbf{c}}(\omega) = \hat{\mathbf{b}}^\dagger(\omega) \cdot \mathcal{G}^{-1}(\omega) \cdot \hat{\mathbf{b}}(\omega) \neq \hat{\mathbf{b}}^\dagger(\omega) \cdot \hat{\mathbf{b}}(\omega). \tag{58}$$

Anyway, because we are working with linear transformations, the most general bilinear form in *Inside* creation operators $\hat{b}_i^\dagger(\omega)$ is still bilinear in *Outside* creation operators $\hat{c}_i^\dagger(\omega)$:

$$\begin{aligned} &\gamma_{11}\hat{c}_1^\dagger\hat{c}_1^\dagger + \gamma_{12}\hat{c}_1^\dagger\hat{c}_2^\dagger + \gamma_{21}\hat{c}_2^\dagger\hat{c}_1^\dagger + \gamma_{22}\hat{c}_2^\dagger\hat{c}_2^\dagger \\ &=\beta_{11}\hat{b}_1^\dagger\hat{b}_1^\dagger + \beta_{12}\hat{b}_1^\dagger\hat{b}_2^\dagger + \beta_{21}\hat{b}_2^\dagger\hat{b}_1^\dagger + \beta_{22}\hat{b}_2^\dagger\hat{b}_2^\dagger. \end{aligned} \tag{59}$$

Because the first of the two forms applied to free space generates a two-photon state of the electromagnetic field, the equality ensures the same for the second one. Therefore, if it is possible to associate to the most general two-photon state generated by *Inside* operators a state with two photons physically generated **inside** the cavity, then, from the relations between *Input* and *Output* operators, we can obtain information about the field outside the cavity, that is the actual object of measurement. In the following section we study how we can do this.

4 States of the Field

We define the states generated by the linear and bilinear forms of *Inside* operators as:

$$\begin{aligned} \hat{b}_i^\dagger(\omega)|0\rangle &\equiv |F_i(\omega); \mathrm{In}\rangle, \\ \hat{b}_i^\dagger(\omega)\hat{b}_j^\dagger(\omega')|0\rangle &\equiv |F_i(\omega), F_j(\omega'); \mathrm{In}\rangle, \end{aligned} \tag{60}$$

where $F_i(\omega)$ is a label which depends on continuous variable ω and on discrete variable $i = 1, 2$. Because of (52) these states are not orthogonal:

$$\langle F_i(\omega); \mathrm{In}|F_j(\omega'); \mathrm{In}\rangle = G_{ij}(\omega)\delta(\omega - \omega'), \tag{61}$$

and

$$\begin{aligned} &\langle F_i(\omega_1), F_j(\omega_2); \mathrm{In}|F_k(\omega_3), F_l(\omega_4); \mathrm{In}\rangle = \\ &\qquad G_{ik}(\omega_1)G_{jl}(\omega_2)\delta(\omega_1 - \omega_3)\delta(\omega_2 - \omega_4) \\ &\qquad + G_{il}(\omega_1)G_{jk}(\omega_2)\delta(\omega_2 - \omega_3)\delta(\omega_1 - \omega_4). \end{aligned} \tag{62}$$

From Eqs. (61-62) we note that anomalous commutation rules, represented by the 2×2 hermitian matrix $\mathcal{G}(\omega)$, form a metric in the two-dimensional Hilbert space generated by *Inside* operators $\hat{b}_i(\omega)$. For example if we write the most general one-photon state created by *Inside* operators as

$$|\phi\rangle = \sum_{i=1}^{2} \int d\omega\, K_i(\omega)|F_i(\omega); \mathrm{In}\rangle, \tag{63}$$

where $K_i(\omega) \in \mathbb{C}$, then its norm is

$$\begin{aligned} \langle\phi|\phi\rangle &= \sum_{i,j}^{1,2} \int d\omega\, K_i^*(\omega)G_{ij}(\omega)K_j(\omega) \\ &= \int d\omega\, \mathbf{K}^\dagger(\omega) \cdot \mathcal{G}(\omega) \cdot \mathbf{K}(\omega), \end{aligned} \tag{64}$$

where it is clear the metric-like role of matrix $\mathcal{G}(\omega)$.
As before, we can define the states generated by the linear and bilinear forms of *Outside* operators as

$$\begin{aligned} \hat{c}_i^\dagger(\omega)|0\rangle &\equiv |F_i(\omega);\mathrm{Out}\rangle, \\ \hat{c}_i^\dagger(\omega)\hat{c}_j^\dagger(\omega')|0\rangle &\equiv |F_i(\omega),F_j(\omega');\mathrm{Out}\rangle, \end{aligned} \tag{65}$$

which, using Eq. (54), can be written in terms of *Inside* operators:

$$\begin{aligned} |F_i(\omega);\mathrm{Out}\rangle &\equiv \sum_k^{1,2} M_{ik}^*(\omega)|F_i(\omega);\mathrm{In}\rangle, \\ |F_i(\omega),F_j(\omega');\mathrm{Out}\rangle &\equiv \sum_{k,l}^{1,2} M_{ik}^*(\omega)M_{jl}^*(\omega')|F_k(\omega),F_l(\omega');\mathrm{In}\rangle, \end{aligned} \tag{66}$$

where $M_{ij}(\omega) \equiv [\mathcal{M}]_{ij}$. Of course these state are orthonormal.
It is possible to make number-states also for a continuous distribution of modes, following Blow*et al.*'s method [21]. On this purpose we define two operators $\hat{C}_i(\eta)$ as

$$\hat{C}_i(\eta) = \int d\omega\, \eta_i^*(\omega)\hat{c}_i(\omega), \qquad i = 1,2\,, \tag{67}$$

where $\eta_i(\omega)$ are two arbitrary complex functions which satisfy the normalization condition.

$$\int d\omega |\eta_i(\omega)|^2 = 1, \qquad i = 1,2. \tag{68}$$

It is easy to verify that these new operators obey the following commutation relations:

$$\left[\hat{C}_i(\eta),\hat{C}_j^\dagger(\eta)\right] = \delta_{ij}, \tag{69}$$

and therefore they can be used to make the number-states by the usual method,

$$|F_i^n(\eta); \text{Out}\rangle = (n!)^{-1/2}[\hat{C}_i^\dagger(\eta)]^n|0\rangle. \tag{70}$$

Using (68) you can verify that the states $|F_i^n(\eta); \text{Out}\rangle$ are correctly normalized:

$$\langle F_i^n(\eta); \text{Out}|F_j^m(\eta); \text{Out}\rangle = \delta_{ij}\delta_{nm}. \tag{71}$$

The construction of number-states is more problematic for *Inside* operators. As before we define

$$\hat{B}_i(\xi) = \int d\omega\, \xi_i^*(\omega)\hat{b}_i(\omega), \qquad i = 1, 2\ , \tag{72}$$

where $\xi_i(\omega)$ are two complex arbitrary functions which can be chosen to satisfy the four conditions

$$\left[\hat{B}_i(\xi), \hat{B}_j^\dagger(\xi)\right] = \int d\omega\, \xi_i^*(\omega)G_{ij}(\omega)\xi_j(\omega) \equiv \Gamma_{ij}(\xi), \tag{73}$$

where $\Gamma_{ij}(\xi)$ is a given matrix. Suppose to set $\Gamma_{ij}(\xi) = \delta_{ij}$. Because $\Gamma(\xi)$ is hermitian by construction, Eq (73) corresponds to $2 \oplus 2$ conditions, two real and a complex one , which the two complex arbitrary functions $\xi_i(\omega)$ $(i = 1, 2)$ must satisfy. The ξ_i modules can be determined imposing $\Gamma_{ii}(\xi) = 1$. In fact, given an arbitrary function $\bar{\xi}(\omega)$ such as

$$\int d\omega |\bar{\xi}(\omega)|^2 = 1, \tag{74}$$

if $G_{ii}(\omega) \neq 0$, that is if $|r_1(\omega)r_2(\omega)|^2 \neq 1$, we can write $\xi_i(\omega)$ as

$$\xi_i(\omega) = \frac{|\bar{\xi}(\omega)|e^{i\phi_i(\omega)}}{\sqrt{G_{ii}(\omega)}}, \qquad i = 1, 2\ , \tag{75}$$

where $\phi_i(\omega)$ is an arbitrary phase, and we can obtain $\Gamma_{ii}(\xi) = 1$. Phases are still arbitrary, but in off-diagonal elements of $\Gamma(\xi)$ there is only the phase difference between $\xi_1(\omega)$ and $\xi_2(\omega)$ which is not sufficient, by itself, to satisfy the two requested conditions. In fact, you can see that only proper linear combinations of *Inside* operators can generate canonical commutation relations. Consider the unitary matrix $\mathcal{U}(\omega)$ that makes $\mathcal{G}(\omega)$ diagonal

$$\mathcal{U}^\dagger(\omega) \cdot \mathcal{G}(\omega) \cdot \mathcal{U}(\omega) = \mathcal{D}(\omega), \tag{76}$$

where the diagonal matrix $\mathcal{D}(\omega)$ has elements $D_{ij}(\omega) = \lambda_i(\omega)\delta_{ij}$, being $\lambda_i(\omega),\ i = 1, 2$ the two $\mathcal{G}(\omega)$'s eigenvalues,

$$\lambda_i(\omega) = G_{11}(\omega) - (-1)^i |G_{12}(\omega)|, \quad i = 1, 2\ . \tag{77}$$

Then if we define the operators $\hat{d}_i(\omega)$ as

$$\hat{\mathbf{d}}(\omega) \equiv \mathcal{U}^\dagger(\omega)\hat{\mathbf{b}}(\omega), \tag{78}$$

you can see that they satisfy the following "quasi-canonical" relations:

$$\left[\hat{d}_i(\omega), \hat{d}_j^\dagger(\omega')\right] = \lambda_i(\omega)\delta_{ij}\delta(\omega - \omega'). \tag{79}$$

If we want to obtain fully canonical relations it is necessary to break the unitariety of the relation between operators $\hat{d}_i(\omega)$ and $\hat{b}_i(\omega)$ introducing the matrix $\mathcal{E}(\omega)$ with elements $E_{ij} = (\lambda_i)^{-1/2}\delta_{ij}$ and to define the operators $\hat{b}'_i(\omega)$ as

$$\hat{\mathbf{b}}'(\omega) \equiv \mathcal{E}(\omega)\hat{\mathbf{d}}(\omega). \tag{80}$$

In fact an easy calculation shows that

$$\left[\hat{b}'_i(\omega), \hat{b}'^\dagger_j(\omega')\right] = \delta_{ij}\delta(\omega - \omega'), \tag{81}$$

where, using Eq. (80),

$$\hat{b}'_j(\omega) \equiv \frac{1}{\sqrt{2\lambda_j(\omega)}} \left[e^{i\phi(\omega)/2}\hat{b}_1(\omega) - (-1)^j e^{-i\phi(\omega)/2}\hat{b}_2(\omega) \right] \qquad j = 1,2\,, \tag{82}$$

and $\phi(\omega) = \arg[G_{12}(\omega)]$ is the relative phase of the two components of $\mathcal{G}(\omega)$ eigenvectors.
Of course by means of these operators we could make orthonormal *Input* number-state but they will result to be associated to functions $\sin(\omega z/c)$ and $\cos(\omega z/c)$ and, as a consequence, the free field-like representation will be lost.

However it is still possible to write the most general two-photon state generated by *Inside* operators $\hat{b}_i(\omega)$ as,

$$|\psi\rangle = \sum_{i,j}^{1,2} \int d\omega \int d\omega' K_{ij}(\omega,\omega') |F_i(\omega), F_j(\omega'); \mathrm{In}\rangle, \tag{83}$$

where, by construction, the matrix $\mathcal{K}(\omega,\omega')$ of elements $K_{ij}(\omega,\omega')$, satisfy

$$\mathcal{K}(\omega,\omega') = \mathcal{K}^{\mathrm{T}}(\omega',\omega), \tag{84}$$

where “T” indicate transposition. In fact the matrix $\mathcal{K}(\omega,\omega')$ is fixed by the emission process inside the cavity.
In a simpler way, a two-photon state generated by *Outside* operators (67), can be written as

$$\begin{aligned} |F_a(\eta), F_b(\eta); \mathrm{Out}\rangle &= (2^{-1/2})^{\delta_{ab}} \hat{C}_a^\dagger(\eta)\hat{C}_b^\dagger(\eta)|0\rangle \\ &= (2^{-1/2})^{\delta_{ab}} \int d\omega \int d\omega' \eta_a(\omega)\eta_b(\omega') |F_a(\omega), F_b(\omega'); \mathrm{Out}\rangle, \end{aligned} \tag{85}$$

where $(a, b = 1,2)$. In the next section we will show how to express this state by *Inside* states.

5 Two-photon states probability distributions

It is well know that the inverse of photon mean flight time in a planar Fabry-Pérot cavity is given by [22]

$$\gamma_{\text{cav}} \cong \frac{c}{l} \frac{1 - |r_1(\omega) r_2(\omega)|}{2|r_1(\omega) r_2(\omega)|^{1/2}}. \tag{86}$$

Now we consider the spontaneous emission of a pair of two identical atoms or molecules within the microcavity [10]. Let γ_{atom} the single atomic decay rate. In the atom-dominate decay regime (that is when $\gamma_{\text{cav}} \ll \gamma_{\text{atom}}$ [23]) , for $1/\gamma_{\text{atom}} \ll t \ll 1/\gamma_{\text{cav}}$ the electromagnetic field can be found in a state like to $|\psi\rangle$. If the matrix $\mathcal{M}(\omega)$ should be unitary, we should calculate easily, as in the quantum theory of a lossless beam-splitter [24], the probability distribution of photon number-states outside the cavity. This is not our case, however we will see that after renormalization of the state $|\psi\rangle$ coefficients $K_{ij}(\omega, \omega')$, it is possible to obtain significant results. For this, we calculate the probability amplitude to find the electromagnetic field, represented by the state $|\psi\rangle$ within the cavity, in the outside state $|F_a(\eta), F_b(\eta); \text{Out}\rangle$, $(a, b = 1, 2)$. It is simple to show with the use of Eqs. (66) and (82)-(85), that result is

$$\begin{aligned} \langle F_a(\eta), F_b(\eta); \text{Out}|\psi\rangle &= 2(2^{-1/2})^{\delta_{ab}} \int d\omega \int d\omega' \eta_a(\omega) \eta_b(\omega') \\ &\times \left[\mathcal{M}(\omega) \cdot \mathcal{G}(\omega) \cdot \mathcal{K}(\omega, \omega') \cdot \mathcal{G}^{\mathrm{T}}(\omega') \cdot \mathcal{M}^{\mathrm{T}}(\omega')\right]_{ab}. \end{aligned} \tag{87}$$

By a lengthy but straightforward calculation, it is simple to show that

$$\left[\mathcal{M}(\omega) \cdot \mathcal{G}(\omega) \cdot \mathcal{K}(\omega, \omega') \cdot \mathcal{G}^{\mathrm{T}}(\omega') \cdot \mathcal{M}^{\mathrm{T}}(\omega')\right]_{ab} \equiv P_{ab}(\omega, \omega'), \tag{88}$$

where we have defined the 2×2 matrix elements $P_{ab}(\omega, \omega')$ as

$$
\begin{aligned}
P_{11}(\omega,\omega') &= \mathcal{L}_2(\omega)\mathcal{L}_1(\omega')\left[K_{11}+K_{22}\alpha_1(\omega)\alpha_1(\omega')+K_{12}\alpha_1(\omega')+K_{21}\alpha_1(\omega)\right],\\
P_{12}(\omega,\omega') &= \mathcal{L}_2(\omega)\mathcal{L}_1(\omega')\left[K_{11}\alpha_2(\omega')+K_{22}\alpha_1(\omega)+K_{12}+K_{21}\alpha_1(\omega)\alpha_2(\omega')\right],\\
P_{21}(\omega,\omega') &= \mathcal{L}_1(\omega)\mathcal{L}_2(\omega')\left[K_{22}\alpha_1(\omega')+K_{11}\alpha_2(\omega)+K_{21}+K_{12}\alpha_2(\omega)\alpha_1(\omega')\right],\\
P_{22}(\omega,\omega') &= \mathcal{L}_1(\omega)\mathcal{L}_2(\omega')\left[K_{22}+K_{11}\alpha_2(\omega)\alpha_2(\omega')+K_{21}\alpha_2(\omega')+K_{12}\alpha_2(\omega)\right],
\end{aligned}
\tag{89}
$$

being

$$
\mathcal{L}_i(\omega)\equiv\frac{t_i(\omega)}{D(\omega)},\qquad \alpha_i(\omega)\equiv r_i(\omega)e^{i\omega l/c},\qquad i=1,2. \tag{90}
$$

Now we evaluate the ratio $\mathcal{R}_{\text{Out}}(R,L|R,R)$ between the probability $P_{\text{Out}}(R,L)$ of observing one photon behind mirror 2 and one photon behind mirror 1 (coincidence), and the probability $P_{\text{Out}}(R,R)$ of observing two photon behind mirror 2.
Using Eqs. (87) we obtain

$$
\begin{aligned}
\mathcal{R}_{\text{Out}}(R,L|R,R) &= \frac{P_{\text{Out}}(R,L)}{P_{\text{Out}}(R,R)} = \left|\frac{\langle F_1(\eta),F_2(\eta);\text{Out}|\psi\rangle}{\langle F_1^2(\eta);\text{Out}|\psi\rangle}\right|^2\\
&= \left|\frac{\displaystyle\int d\omega_1\int d\omega_2\,\eta_1^*(\omega_1)\eta_2^*(\omega_2)P_{12}(\omega_1,\omega_2)}{2^{-1/2}\displaystyle\int d\omega_1\int d\omega_2\,\eta_1^*(\omega_1)\eta_1^*(\omega_2)P_{11}(\omega_1,\omega_2)}\right|^2.
\end{aligned}
\tag{91}
$$

We now illustrate the meaning of this formula. We start writing explicitally the value of the ratio P_{12}/P_{11}:

$$
\frac{P_{12}(\omega_1,\omega_2)}{P_{11}(\omega_1,\omega_2)}=\frac{K_{11}r_2(\omega_2)e^{i\omega_2 l/c}+K_{22}r_1(\omega_1)e^{i\omega_1 l/c}+K_{12}+K_{21}r_1(\omega_1)e^{i\omega_1 l/c}r_2(\omega_2)e^{i\omega_2 l/c}}{K_{11}+K_{22}r_1(\omega_1)e^{i\omega_1 l/c}r_1(\omega_2)e^{i\omega_2 l/c}+K_{12}r_1(\omega_2)e^{i\omega_2 l/c}+K_{21}r_1(\omega_1)e^{i\omega_1 l/c}}. \tag{92}
$$

This expression is only apparently complicated, but each term at numerator and denominator is susceptible to a clear physical interpretation. We assume that $K_{ij}(\omega_1,\omega_2)$ is proportional to the probability amplitude that a pair of excited molecules within the cavity, emit spontaneously one photon with angular frequency ω_1 on mode i, and one photon with angular frequency ω_2 on mode j, being the proportionality factor the same for all coefficients $K_{ij}(\omega_1, \omega_2)$. More precisely we assume that $|K_{ij}(\omega_1, \omega_2)|^2 d\omega_1 d\omega_2$ is proportional to the emission probability of one photon on mode i with angular frequency between ω_1 and $\omega_1 + d\omega_1$, and one photon on mode j with angular frequency between ω_2 and $\omega_2 + d\omega_2$. At this point it is easy to see how each term which appears in the ratio (92) admits a clear physical interpretation.

$$K_{12} + K_{21} r_1(\omega_1) e^{i\omega_1 l/c} r_2(\omega_2) e^{i\omega_2 l/c}$$

$$K_{11} r_2(\omega_2) e^{i\omega_2 l/c} + K_{22} r_1(\omega_1) e^{i\omega_1 l/c}$$

$$K_{11} + K_{22} r_1(\omega_1) e^{i\omega_1 l/c} r_1(\omega_2) e^{i\omega_2 l/c}$$

$$K_{12} r_1(\omega_2) e^{i\omega_2 l/c} + K_{21} r_1(\omega_1) e^{i\omega_1 l/c}$$

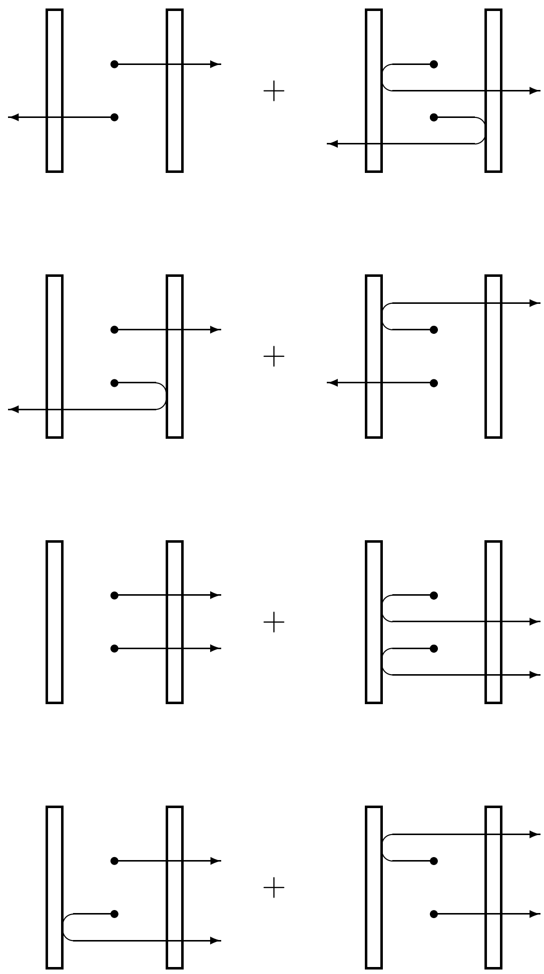

FIG. 5. Diagrams illustrating the probability amplitudes (reported in the left column), relative to Eq. (92) Here $r_1(\omega)$ [$r_2(\omega)$] is the reflection coefficient of mirror 1 (at the left) [2 (at the right)]. The photon of angular frequency ω_1 is always plot higher then photon of angular frequency ω_2.

With the help of Fig. 5 we can see, e.g., that the first term at numerator, corresponding to the first diagrams of the second row, give us the probability amplitude of simultaneous emission of a pair of photons toward right, and that the photon of angular frequency ω_1 is detected behind the mirror 2, while the photon of angular frequency ω_2 is detected behind the mirror 1 after reflection on the mirror 2. The transmission coefficients and all contributes generated by multiple reflections on the cavity mirrors, are computed into terms $\mathcal{L}_2(\omega_1)\mathcal{L}_1(\omega_2)$, which we have simplified into Eq. (92). All the other terms in Eq. (92), admit analogue interpretation shown by remaining diagrams in Fig. 5.
Of course, for reasons of internal consistency of the theory, it need renormalize the coefficients $K_{ij}(\omega_1, \omega_2)$ imposing

$$\sum_{i,j}^{1,2} \int d\omega_1 \int d\omega_2 |K_{ij}(\omega_1, \omega_2)|^2 = 1. \tag{93}$$

At this point it is easy to obtain the correct probability distributions. If we define

$$\begin{aligned} \mathcal{R}_{\mathrm{Out}}(R, L|R, R) &\equiv \mathcal{R}_1, \\ \mathcal{R}_{\mathrm{Out}}(R, L|L, L) &\equiv \mathcal{R}_2, \end{aligned} \tag{94}$$

and we impose the normalization condition

$$P_{\mathrm{Out}}(R, R) + P_{\mathrm{Out}}(R, L) + P_{\mathrm{Out}}(L, L) = 1, \tag{95}$$

the desired distributions are then obtained after some algebra in the form

$$\begin{aligned} P_{\mathrm{Out}}(R, R) &= \frac{\mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_1\mathcal{R}_2}, \\ P_{\mathrm{Out}}(R, L) &= \frac{\mathcal{R}_1\mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_1\mathcal{R}_2}, \\ P_{\mathrm{Out}}(L, L) &= \frac{\mathcal{R}_1}{\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_1\mathcal{R}_2}, \end{aligned} \tag{96}$$

where, for example, $P_{\mathrm{Out}}(R, R)$ is the normalized probability to find two photon outside the cavity behind the mirror 2 of the cavity.

5.1 Single mode

Now we suppose that the field mode spectrum is discretized by an appropriate procedure [21], furthermore we fix the attention on a single mode of assigned angular frequency ω. The commutation relations for creation and annihilation operators defined on this discrete set of modes, are written as

$$[\hat{a}_i, \hat{a}_j^\dagger] = [\hat{c}_i, \hat{c}_j^\dagger] = \delta_{ij}, \tag{97}$$

$$[\hat{b}_i, \hat{b}_j^\dagger] = G_{ij}. \tag{98}$$

Now we can have two photon on a single discrete mode, therefore we define the states generated by bilinear and quadratic forms of *Inside* and *Outside* operators, as

$$|f_i, f_j; \mathrm{In}\rangle \equiv \left(2^{-1/2}\right)^{\delta_{ij}} \hat{b}_i^\dagger \hat{b}_j^\dagger |0\rangle, \tag{99}$$

and

$$\begin{aligned} |f_i, f_j; \mathrm{Out}\rangle &\equiv \left(2^{-1/2}\right)^{\delta_{ij}} \hat{c}_i^\dagger \hat{c}_j^\dagger |0\rangle \\ &= \left(2^{-1/2}\right)^{\delta_{ij}} \sum_{k,l}^{1,2} \left(2^{-1/2}\right)^{-\delta_{kl}} M_{ik}^* M_{jl}^* |f_k, f_l; \mathrm{In}\rangle. \end{aligned} \tag{100}$$

respectively. It is easy to see that

$$\langle f_i, f_j; \mathrm{In}|f_k, f_l; \mathrm{In}\rangle = \left(2^{-1/2}\right)^{\delta_{ij}+\delta_{kl}} \left(G_{ik}G_{jl} + G_{il}G_{jk}\right). \tag{101}$$

Exactly as before, we define the most general two-photon state created by *Inside* operators as

$$|\psi\rangle = \sum_{i,j}^{1,2} K_{ij} |f_i, f_j; \mathrm{In}\rangle, \tag{102}$$

where $\mathcal{K} = \mathcal{K}^{\mathrm{T}}$, and we calculate

$$\langle f_a, f_b; \mathrm{Out}|\psi\rangle = 2\left(2^{-1/2}\right)^{\delta_{ab}} \left(\mathcal{M} \cdot \mathcal{G} \cdot \overline{\mathcal{K}} \cdot \mathcal{G}^{\mathrm{T}} \cdot \mathcal{M}^{\mathrm{T}}\right)_{ab}, \tag{103}$$

where we have defined the matrix $\overline{\mathcal{K}}$ as

$$[\overline{\mathcal{K}}]_{ij} \equiv \overline{K}_{ij} = \left(2^{-1/2}\right)^{\delta_{ij}} K_{ij}. \tag{104}$$

Therefore Eq. (89) is formally still valid if we make the substitution $K_{ii} \to K_{ii}/\sqrt{2}$. From this point we consider the case of a symmetrical cavity, that is we assume $r_1(\omega) = r_2(\omega) \equiv r(\omega)$ and $t_1(\omega) = t_2(\omega) \equiv t(\omega)$. For simplicity we choose the phase of transmission and reflection coefficients, as

$$t(\omega) = i\sqrt{1-R} \qquad \text{and} \qquad r(\omega) = -\sqrt{R}, \tag{105}$$

and redefine, into a more expressive form,

$$K_{11} \equiv C_{RR}, \qquad K_{22} \equiv C_{LL}, \qquad K_{12} + K_{21} = 2K_{12} \equiv C_{RL}. \tag{106}$$

Then in discrete mode representation, the normalization conditions, can be written as

$$|C_{RR}|^2 + |C_{RL}|^2 + |C_{LL}|^2 = 1. \tag{107}$$

Finally we can write:

$$\mathcal{R}_{\mathrm{Out}}(R, L|R, R) = \left| \frac{\sqrt{2}(C_{RR} + C_{RR}) r e^{i\omega l/c} + C_{RL}(1 + r^2 e^{2i\omega l/c})}{C_{RR} + C_{LL} r^2 e^{2i\omega l/c} + \sqrt{2} C_{RL} r e^{i\omega l/c}} \right|^2. \tag{108}$$

We note the presence of factors $\sqrt{2}$, which we have introduced because of different

normalization required by discrete mode spectrum. Because of them, there is no exact correspondence between diagram of Fig. 5 and terms into Eq. (108). This factors arise since we are not using an orthonormal base and therefore a mixing between normalization of states with a photon on mode and two photons on mode, is produced. Indeed we will see, in the next section, that working with single photon states which admit only a single normalization factor, it is possible to obtain a direct association between diagrams and formulas.
Now we calculate Eq. (108) for some particular value of the state $|\psi\rangle$. Let $|\psi\rangle$ coinciding with each of the three states of the orthonormal base defined in Appendix. It can readily be shown that

a) $|\psi\rangle = |n_\pm\rangle$

$$C_{RL} = \pm\frac{1}{\sqrt{2}}, \qquad C_{RR} = C_{LL} = \frac{1}{2} \qquad \Rightarrow \qquad \mathcal{R}_{\mathrm{Out}}(R, L|R, R) = 2. \tag{109}$$

b) $|\psi\rangle = |n_0\rangle$

$$C_{RL} = 0, \qquad C_{RR} = -C_{LL} = -\frac{1}{\sqrt{2}} \qquad \Rightarrow \qquad \mathcal{R}_{\mathrm{Out}}(R, L|R, R) = 0. \tag{110}$$

It is remarkable that for these states having high symmetry, $\mathcal{R}_{\mathrm{Out}}$ does not depend either on mirrors reflectivity, nor on the phase $\omega l/c$. Following our interpretative scheme, Eq. (109) shows that when the emission probability of the pair of photons on the same way or on the opposite way, is the same, that is $|C_{RL}|^2 = |C_{RR}|^2 + |C_{LL}|^2 = 1/2$, the probability to observe a coincidence is twice with respect to the probability of not observing. Instead when the two photon are emitted along a common way, but with pair emission probability amplitude toward right and left which differs for a sign, Eq. (110), the probability of observing a coincidence is zero. But within our interpretative scheme, which require none distinction between left and right for emission of the pair of the photon in a symmetrical cavity, it is hard to think that this state really exist. Therefore a state $|n_0\rangle$ having $C_{RR} \neq C_{LL}$ is difficult to accept. From this point we consider only the case $C_{RR} = C_{LL}$ and we define

$$\frac{C_{RR}}{C_{RL}} \equiv \zeta = |\zeta| e^{i \arg \zeta}. \tag{111}$$

Then using Eq. (111), Eq. (108) can be written as

$$\mathcal{R}_{\mathrm{Out}}(R,L|R,R) = \tag{112}$$

$$\frac{8R^2z^2 - 4\sqrt{2R}\,[\cos(x+y) + R\cos(x-y)]\,z + 1 + 2R\cos 2x + R^2}{(1+2R\cos 2x + R^2)\,z^2 - 2\sqrt{2R}\,[\cos(x-y) + R\cos(x+y)]\,z + 2R},$$

where we have used the following notation

$$x \equiv \omega l/c, \qquad y \equiv \arg\zeta, \qquad z \equiv |\zeta|. \tag{113}$$

This expression seems still rather complicated, but we can learn something from it considering the limit cases $C_{RR} = 0 \Leftrightarrow z = 0$ and $C_{RL} = 0 \Leftrightarrow z = \infty$:

$$\begin{aligned} \lim_{z\to 0} \mathcal{R}_{\mathrm{Out}}(R,L|R,R) &= \frac{1+2R\cos 2x + R^2}{2R} \equiv \mathcal{R}_0, \\ \lim_{z\to \infty} \mathcal{R}_{\mathrm{Out}}(R,L|R,R) &= \frac{8R^2}{1+2R\cos 2x + R^2} \equiv \mathcal{R}_\infty. \end{aligned} \tag{114}$$

The first of Eqs. (114) is shown in Fig. 6. From Fig. 6b we can see that when $R \to 0$, $\mathcal{R}_0 \to \infty$ that is, from Eq. (108), $P_{\mathrm{Out}}(R,R) \to 0$. Indeed if in the absence of the cavity the two of photon are emitted one toward right and one toward left, it is impossible to detect two from the same side. When $R \gtrsim 0.8$, $\mathcal{R}_0$ is practically independent from R while it presents an oscillation of period π in x. Observing Fig. 6a is evident that when $R \gtrsim 0.5$, near the resonance $x \simeq \pi$ we have $\mathcal{R}_0 \simeq 2$, while near $x = (2n+1)\pi/2$, n integer, there is a region for which $\mathcal{R}_0 < 1$. This loss of coincidence is due to the fact that only the first and the last pairs of diagrams in Fig. 5. contribute to $\mathcal{R}_0$ but in the first, which gives the probability amplitude of observing the coincidence, the two amplitudes C_{RL} and $C_{RL}Re^{2i\omega l/c}$ interfere destructively for $x \simeq (2n+1)\pi/2$ and $R \simeq 1$ causing zero total amplitude. The second of Eqs. (114) is shown in Fig. 7a for $0 < \mathcal{R}_0 < 1$; the plane part of the graph corresponds to the bigger than 1 values. First of all we observe an obvious fact: for $R = 0$, we have $\mathcal{R}_\infty = 0$, that is if the two photons are emitted both on the same way, it is impossible to observe a coincidence in absence of a cavity that mixes the directions. From the graph it is evident as when R increment, the reflections increment too,

and the coincidence probability arise from the zero value. In Fig. 7b the behaviour of $\mathcal{R}_\infty$ is shown for $\mathcal{R}_\infty \leq 50$. We note that when $x = (2n+1)\pi/2$ and $R \to 1$, we have $\mathcal{R}_\infty \to \infty$, that is the probability of observing a pair of photons from a side of the cavity, goes to zero. Indeed the second and the third diagram of Fig. 5 contribute to $\mathcal{R}_\infty$ but the third diagram, giving the probability of observation two parallel photons, goes to zero for $x = (2n+1)\pi/2$ and $R \simeq 1$ because of destructive interference between C_{RR} and $C_{LL}Re^{2i\omega l/c}$. Therefore for $z \gg 1$ and realistic reflectivity, the probability of observing coincidence is always bigger then probability of observing two photon on the same side of the cavity.
On the other hand, we have also seen that for $z \to 0$, $\mathcal{R}_{\mathrm{Out}}$ can be less then 1 for $R \to 1$ and this is certainly the most interesting case to investigate. In Fig. 8 we have shown the behaviour of Eq. (112), as function of x and y, for several values z and $R = .999$. For $R \gtrsim 0.8$ the dependence from R is negligible. In Fig. 9 we shown the contour plot of $\mathcal{R}_{\mathrm{Out}}$, between 0 and 1. From this figure it is evident that the dark zones, corresponding to $\mathcal{R}_{\mathrm{Out}} < 1$, have an extension gradually decreasing for z increasing, until they disappear for $z \geq 1$ (not shown in Fig. 9). It is interesting to note that the probability of observing two photon on one side of the cavity is bigger than coincidence probability, when within the cavity the two photons are emitted along opposite way. Indeed only third and fourth diagrams in Fig. 5 contribute to $P_{\mathrm{Out}}(R,R)$, but while in the fourth diagram the two amplitudes are always in phase, in the third diagram the two amplitude C_{RR} and $C_{LL}Re^{2i\omega l/c}$ can have opposite phase and interfere destructively for $R \sim 1$. Then if $|C_{RR}| > |C_{RL}|$, that is if $z > 1$, either the third or the fourth diagram, give negligible contributions and $P_{\mathrm{Out}}(R,R) \sim 0$. Instead if $|C_{RL}| > |C_{RR}|$ $(z < 1)$ the fourth diagram gives a consistent contribution to $P_{\mathrm{Out}}(R,R)$ and at the same time the first diagram (proportional to C_{RL} but in condition of destructive interference) gives a negligible contribution to $P_{\mathrm{Out}}(R,L)$.
Another interesting case is that of the resonance in a broad sense, that is $\omega l/c = \pi N$, with N integer: for N odd there is resonance in a strict sense while for N even there is anti-resonance. In this case Eq. (112) can be simplified and written as

$$\mathcal{R}_{\mathrm{Out}}(R,L|R,R) = \frac{4F^2z^2 - (-1)^N 4Fz\cos y + 1}{z^2 - (-1)^N 2Fz\cos y + F^2}, \tag{115}$$

where $F \equiv \sqrt{2R}/(1+R)$. In the Fig. 10a a plot of Eq. (115) is shown as function of z and y, for $R = .5$ and N odd. The analogue plot for N even can be obtained translating the plot by an amount π along the y axes. It is evident that the ratio $\mathcal{R}_{\mathrm{Out}}$ is always greater then 1 except for a small region centered around $y = \pi$ and $z = 1/2F$ which disappear for $R \to 1$. We can always write Eq. (115) as a ratio between two second degree polynomial in z

$$\mathcal{R}_{\rm Out}(R,L|R,R) = 4F^2 \frac{(z-z_+^u)(z-z_-^u)}{(z-z_+^d)(z-z_-^d)}, \tag{116}$$

where we have defined the roots of the two polynomials as

$$z_\pm^u \equiv -\frac{1}{2F} e^{\pm iy}, \qquad z_\pm^d \equiv -F e^{\pm iy}. \tag{117}$$

Only for $y = \pi \,(\mathrm{mod}\, 2\pi)$ we can have real positive root:

$$\mathcal{R}_{\rm Out}(R,L|R,R)\Big|_{y=\pi} = \left(\frac{2Fz-1}{z-F}\right)^2. \tag{118}$$

In Fig. 10b a plot of Eq. (118) is shown for values of z near to $1/2F$; from this plot we can see in detail the "jump" from the pole to the zero. From Fig. 10d we can observe that for $R \to 1$ the pole in F and the zero in $1/2F$ tend to the common value $1/\sqrt{2}$ compensating each other, so that $\mathcal{R}_{\rm Out} = 2$. The distance between the pole and the zero decreases as $\sim (1-R)^2$ and already for $R = .9$ is less than a part on a hundred. It is reasonable to think that for higher and more realistic reflectivity, it is not possible to generate really a state so well defined to discriminate between the pole and the zero. Furthermore the really physical situation is always described by a continuous superposition of modes, therefore we think that effective value of $\mathcal{R}_{\rm Out}(R,L|R,R)\Big|_{y=\pi}$ is ~ 2 in all plane $y-z$. At last we note that when $C_{RR} = 0$ or $C_{RL} = 0$, we have respectively

$$\lim_{z\to 0} \mathcal{R}_{\rm Out}(R,L|R,R) = \frac{1}{F^2} \xrightarrow{R\to 1} 2, \tag{119}$$

$$\lim_{z\to \infty} \mathcal{R}_{\rm Out}(R,L|R,R) = 4F^2 \xrightarrow{R\to 1} 2. \tag{120}$$

From Eqs. (119-120) we deduce that in these conditions is physically indifferent if the two photons are emitted in the same or in the opposite way within the cavity. We have already obtained the result $\mathcal{R}_{\rm Out} = 2$, when $|\psi\rangle = |n_\pm\rangle$ independently from R, corresponding to equal probability of emission of a pair of photon along the same way or in opposite way. In the present case we have the same result for $\omega l/c = \pi N$ and $R \sim 1$. This is consistent with the fact that in the limit of total reflectivity in which all frequencies satisfy the resonance (in a broad sense) conditions $\omega_n = n\pi c/l$, n integer, it is impossible to speak of direction (left or right) of emission of a photon, because of the two counter-propagating wave that constitute a stationary wave within the cavity have exactly the same weight. Finally we note that since $P_{\rm Out}(R,R) = P_{\rm Out}(L,L)$, from Eqs (95-96) follows

$$P_{\rm Out}(R,L) = \frac{\mathcal{R}_{\rm Out}}{2+\mathcal{R}_{\rm Out}}, \qquad P_{\rm Out}(R,R) = \frac{1}{2+\mathcal{R}_{\rm Out}}. \tag{121}$$

Then when $\mathcal{R}_{\rm Out} = 2$ we have

$$P_{\rm Out}(R,L) = \frac{1}{2}, \qquad P_{\rm Out}(R,R) = \frac{1}{4}, \tag{122}$$

in qualitative agreement with Ref. [10].

5.2 Single photon states

In this section, we started with investigation of two-photon states, because we were interested at the leak of symmetry in the photon-number probability distributions. Nevertheless the study of one-photon states is not void of interest. Indeed in previous subsection we have shown that in discrete mode representation, the interpretation of the results, were "contaminated" from factors $\sqrt{2}$ generated by mixing between normalization of states with one photon for mode (e.g. $|1,1\rangle$) and two photons for mode (e.g. $|2,0\rangle$). Now we will see that working with one-photon states this mixing never appear. In the most general form, the one-photon state $|\phi\rangle$ generated by *Inside* operators, can be written as

$$|\phi\rangle = \sum_{i=1}^{2} \int d\omega\, K_i(\omega) |F_i(\omega); {\rm In}\rangle, \tag{123}$$

while the analogous state generated by *Outside* operators, is given by

$$
\begin{aligned}
|F_a(\eta);\mathrm{Out}\rangle &= \hat{C}_a^\dagger(\eta)|0\rangle \\
&= \int d\omega\, \eta_a(\omega)|F_a(\omega);\mathrm{Out}\rangle.
\end{aligned} \tag{124}
$$

The probability amplitude to find the electromagnetic field, represented by the state $|\phi\rangle$ within the cavity, in the state $|F_a(\eta);\mathrm{Out}\rangle$, is

$$
\langle F_a(\eta);\mathrm{Out}|\phi\rangle = \sum_{i=1}^{2} \int d\omega\, \eta_a^*(\omega) \left[\mathcal{M}(\omega)\mathcal{G}(\omega)\right]_{ai} K_i(\omega), \tag{125}
$$

The ratio between the probability $P_{\mathrm{Out}}(R)$ of observing a photon behind mirror 2 and the probability $P_{\mathrm{Out}}(L)$ of observing a photon behind mirror 1 is equal to

$$
\begin{aligned}
\mathcal{R}_{\mathrm{Out}}(R|L) = \frac{P_{\mathrm{Out}}(R)}{P_{\mathrm{Out}}(L)} &= \left|\frac{\langle F_1(\eta);\mathrm{Out}|\phi\rangle}{\langle F_2(\eta);\mathrm{Out}|\phi\rangle}\right|^2 \\
&= \left|\frac{\int d\omega\, \eta_1^*(\omega)\mathcal{L}_2(\omega)\left[C_R(\omega) + C_L(\omega)r_1(\omega)e^{i\omega l/c}\right]}{\int d\omega\, \eta_2^*(\omega)\mathcal{L}_1(\omega)\left[C_L(\omega) + C_R(\omega)r_2(\omega)e^{i\omega l/c}\right]}\right|^2,
\end{aligned} \tag{126}
$$

where we have redefined $K_1(\omega) \equiv C_R(\omega)$ and $K_2(\omega) \equiv C_L(\omega)$. Exactly as in section 5.2, if we assume $C_R(\omega)$ ($C_L(\omega)$) proportional to the probability amplitude that an active medium within the cavity emit a photon of angular frequency ω toward right (left), each terms into Eq. (126) admit a clear physical interpretation illustrated in Fig. 11.

Since in discrete mode representation $|f_i;\mathrm{In}\rangle \equiv \hat{b}_i^\dagger|0\rangle$, it is evident that passing to discrete case each term between square bracket in Eq. (126), remain formally unchanged, without any $\sqrt{2}$ factor.

For sake of consistency now we must impose

$$
\int d\omega\, \left\{|C_R(\omega)|^2 + |C_L(\omega)|^2\right\} = 1. \tag{127}
$$

$$C_R(\omega) + C_L(\omega) r_1(\omega) e^{i\omega l/c}$$

$$C_R(\omega) r_2(\omega) e^{i\omega l/c} + C_L(\omega)$$

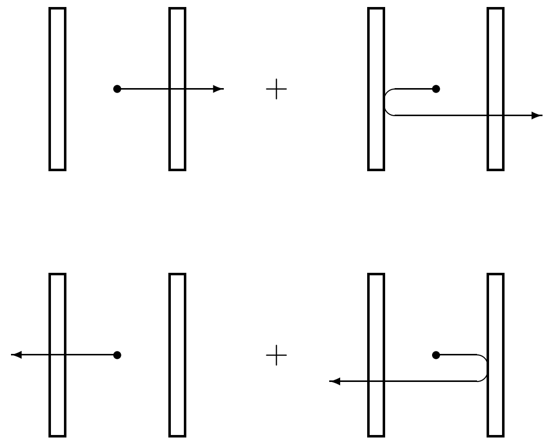

FIG. 11. Diagrams illustrating the probability amplitudes (reported in the left column), relative to Eq. (126) Here $r_1(\omega)$ $[r_2(\omega)]$ is the reflection coefficient of mirror 1 (at the left) [2 (at the right)].

In this case the normalized probability $P_{\rm Out}(R)$ and $P_{\rm Out}(L)$ are given by

$$P_{\rm Out}(R) = \frac{\mathcal{R}_{\rm Out}(R|L)}{1 + \mathcal{R}_{\rm Out}(R|L)}, \qquad P_{\rm Out}(L) = \frac{1}{1 + \mathcal{R}_{\rm Out}(R|L)}. \tag{128}$$

6 Conclusion

We have derived some simple relations for electromagnetic field inside and outside an optical cavity, using a non-unitary transformation between *Inside* and *Outside* operators. The convenience of this approach lies in the fact that we do not need to know any details of internal processes that generate the two photon, for calculate the photon-number probability distribution outside the cavity. Conversely we can obtain information on internal processes, by comparing the calculate and measured probability distribution. The method is a natural extension to non-unitary transformation, of the usual method employed, e.g., in the quantum theory of the lossless Beam-Splitter. [24].

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Appendix

In the section 5.2 we introduced the three orthonormal states $|n_\pm\rangle$ e $|n_0\rangle$ without derivation. This will be done in this Appendix.
We rewritten the commutation relation for operators $\hat{b}_1(\omega)$ and $\hat{b}_2(\omega)$ in discrete mode representation and symmetrical cavity, as

$$\begin{aligned}\left[\hat{b}_2(\omega), \hat{b}_2^\dagger(\omega)\right] &= \left[\hat{b}_1(\omega), \hat{b}_1^\dagger(\omega)\right] \equiv \Delta(\omega), \\ \left[\hat{b}_2(\omega), \hat{b}_1^\dagger(\omega)\right] &= \left[\hat{b}_1(\omega), \hat{b}_2^\dagger(\omega)\right]^* \equiv \rho(\omega)\Delta(\omega),\end{aligned} \tag{129}$$

where we have defined

$$\begin{aligned}\Delta(\omega) \equiv& \frac{1-R^2}{1-2R\cos(2\omega l/c)+R^2}, \\ \rho(\omega) \equiv& -2\frac{\sqrt{R}}{1+R}\cos(\omega l/c).\end{aligned} \tag{130}$$

Using a slightly different notation with respect to section 5.2, we define

$$|\mathcal{R}, \mathcal{L}; \mathrm{In}\rangle \equiv \frac{(\hat{b}_1^\dagger)^\mathcal{R}}{\sqrt{\mathcal{R}!}}\frac{(\hat{b}_2^\dagger)^\mathcal{L}}{\sqrt{\mathcal{L}!}}|0\rangle, \qquad \mathcal{R}+\mathcal{L}=2, \tag{131}$$

where again the factor $(\mathcal{R}!\mathcal{L}!)^{-1/2}$ is due to the possible presence of two photon on a single mode. Of course, since $\hat{b}_1$ and $\hat{b}_2^\dagger$ does not commute, this kets not form an orthonormal base, but they are however linearly independent. Indeed if we define

$$\langle\mathcal{R}, \mathcal{L}; \mathrm{In}|\mathcal{R}', \mathcal{L}'; \mathrm{In}\rangle \equiv \tilde{G}(\mathcal{R}, \mathcal{L}; \mathcal{R}', \mathcal{L}'), \tag{132}$$

we can calculate, using Eqs. (129-130),

$$\tilde{G}(\mathcal{R},\mathcal{L};\mathcal{R}',\mathcal{L}') = \Delta^2 \begin{pmatrix} 1 & \sqrt{2}\rho & \rho^2 \\ \sqrt{2}\rho & 1+\rho^2 & \sqrt{2}\rho \\ \rho^2 & \sqrt{2}\rho & 1 \end{pmatrix}. \tag{133}$$

Therefore the kets defined in Eq. (131) are linearly independent being their Gram determinant positive [25]:

$$\mathrm{Det}[\tilde{G}] = \Delta^6(1-\rho^2)^3 \geq 0. \tag{134}$$

By diagonalization of $\tilde{G}$, after some algebra we obtain the orthonormal base we look for:

$$\begin{aligned} |n_+\rangle &= \frac{1}{2}|2,0;\mathrm{In}\rangle + \frac{1}{\sqrt{2}}|1,1;\mathrm{In}\rangle + \frac{1}{2}|0,2;\mathrm{In}\rangle, \\ |n_0\rangle &= -\frac{1}{\sqrt{2}}|2,0;\mathrm{In}\rangle + \frac{1}{\sqrt{2}}|0,2;\mathrm{In}\rangle, \\ |n_-\rangle &= \frac{1}{2}|2,0;\mathrm{In}\rangle - \frac{1}{\sqrt{2}}|1,1;\mathrm{In}\rangle + \frac{1}{2}|0,2;\mathrm{In}\rangle. \end{aligned} \tag{135}$$

Note on matrix $\tilde{G}$

The form of the matrix $\tilde{G}$ is particular and justify this little note. Let $\mathbf{S}_0$, $\mathbf{S}_1$ and $\mathbf{S}_2$ the following 3×3 matrix:

$$\mathbf{S}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{S}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{S}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{136}$$

It can be readily shown that they satisfy the following multiplication table:

$$\begin{array}{c|ccc} \mathbf{S}_\mu \cdot \mathbf{S}_\nu & \mathbf{S}_0 & \mathbf{S}_1 & \mathbf{S}_2 \\ \hline \mathbf{S}_0 & \mathbf{S}_0 & \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_1 & \mathbf{S}_1 & \mathbf{S}_0 & \mathbf{S}_2 \\ \mathbf{S}_2 & \mathbf{S}_2 & \mathbf{S}_2 & \mathbf{S}_0 + \mathbf{S}_1 \end{array} \tag{137}$$

Now consider the generic matrix $\mathcal{N}(\alpha)$, $(\alpha \in \mathbb{R})$ given by

$$\mathcal{N}(\alpha) = \mathbf{S}_0 + \alpha^2 \mathbf{S}_1 + \sqrt{2}\alpha \mathbf{S}_2. \tag{138}$$

It is characterized by

$$\mathrm{Det}[\mathcal{N}(\alpha)] = (1-\alpha^2)^3, \qquad \mathrm{Tr}[\mathcal{N}(\alpha)] = 3 + \alpha^2. \tag{139}$$

If we indicate with $\lambda_0, \lambda_\pm$ his eigenvalues and with $\mathbf{n}_0, \mathbf{n}_\pm$ the corresponding eigenvectors, we can write

$$\begin{cases} \mathbf{n}_+ = \frac{1}{2}(1, \sqrt{2}, 1) & \lambda_+ = (1+\alpha)^2, \\ \mathbf{n}_0 = \frac{1}{\sqrt{2}}(-1, 0, 1) & \lambda_0 = (1-\alpha^2), \\ \mathbf{n}_- = \frac{1}{2}(1, -\sqrt{2}, 1) & \lambda_- = (1-\alpha)^2. \end{cases} \tag{140}$$

Using Eq. (137) it is easy to see that

$$\mathcal{N}(\alpha) \cdot \mathcal{N}(\beta) = (1+\alpha\beta)^2 \mathcal{N}\left(\frac{\alpha+\beta}{1+\alpha\beta}\right). \tag{141}$$

Since $\mathcal{N}(0) = \mathbf{S}_0$ is the identity matrix, it is clear that the inverse of $\mathcal{N}(\alpha)$ is still a matrix of the form (138). Indeed putting $\beta = -\alpha$ into Eq. (141) we obtain

$$\mathcal{N}(\alpha) \cdot \mathcal{N}(-\alpha) = (1-\alpha^2)^2 \mathbf{S}_0, \tag{142}$$

that is

$$\mathcal{N}^{-1}(\alpha) = \frac{1}{(1-\alpha^2)^2}\mathcal{N}(-\alpha). \tag{143}$$

Finally, from Eq. (133) we get

$$\tilde{G} = \Delta^2 \mathcal{N}(\rho). \tag{144}$$

As a curiosity, we note that in two dimension the matrices $\mathcal{N}(\alpha)$ which satisfy the algebra (141) are given by

$$\mathcal{N}(\alpha) \equiv \sigma_0 + \alpha\sigma_1, \tag{145}$$

where $\sigma_0 = \mathbb{I}$ and σ_1 is the first of the Pauli matrices [26, p. 160]. It is easy to show that

$$\mathrm{Det}[\mathcal{N}(\alpha)] = 1 - \alpha^2, \qquad \mathrm{Tr}[\mathcal{N}(\alpha)] = 2. \tag{146}$$

The eigenvalues $\lambda_\pm$ and the corresponding eigenvectors $\mathbf{n}_\pm$ are given by

$$\begin{cases} \mathbf{n}_+ = \frac{1}{2}(1,1) & \lambda_+ = 1+\alpha, \\ \\ \mathbf{n}_- = \frac{1}{2}(-1,1) & \lambda_- = 1-\alpha. \end{cases} \tag{147}$$

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FIGURE CAPTIONS

FIG. 6. Plots of $\mathcal{R}_0$ for different range of values. In (a) the plane part correspond to values of $\mathcal{R}_0$ greater than 2.5.

FIG. 7. Plots of $\mathcal{R}_\infty$ for different range of values. In (a) the plane part correspond to values of $\mathcal{R}_\infty$ greater than 1.

FIG. 8. Four plot of $\mathcal{R}_{\mathrm{Out}}$ calculated for a symmetrical cavity, $R = 0.999$ and several values of z. The dependence from R for $R \gtrsim 0.8$ is negligible and not reported in the figure.

FIG. 9. Contour plot of corresponding plot in Fig. 8, shows for values $\mathcal{R}_{\mathrm{Out}}$ between 0 and 1. It is evident that for z increasing, the zone on the plane $x - y$ in which $\mathcal{R}_{\mathrm{Out}} < 1$ decreasing.

FIG. 10. (a) Plot of Eq. (115) for $R = 0.5$. This is not a realistic value, but we choose it for show in a clear manner the various quantities. (b) The same that in (a) for $y = \pi$; the vertical straight lines passing through the pole in F and through the zero in $1/2F$. (c) The same that in (a) for $z = 1/2F$. (d) Plot of $1/2F$ (up) and F (down), as function of the reflectivity R.